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LETTER TO THE EDITOR

Hamiltonian of a 1D quantum chain for Belavin's $\mathbb{Z}_n \times \mathbb{Z}_n$ model

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Abstract. The Hamiltonian of a 1D quantum chain corresponding to Belavin's $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric model is derived. This Hamiltonian is a multicomponent generalisation of the XYZ model with n^2 coupling constants in general. In the limit $\tau \to i\infty$ it reduces to the generalised XXZ model with $(n-1)$ n-fold-degenerate coupling constants and n non-degenerate ones. The Hamiltonian is Hermitian only for $n=2$ or for $n>2$ with a crossing parameter w restricted to integers. There exists a domain of parameters in which the Boltzmann weights are positive but the corresponding Hamiltonian is non-Hermitian. The relations of the Hamiltonian to other models are discussed.

1. Introduction

An important discovery of exactly solvable models is the connection between 2D statistical models and the Heisenberg 1D quantum chains [1–7]. In 1970 Sutherland showed that the transfer matrix of a zero-field eight-vertex model commutes with the Hamiltonian of the XYZ model [3]. In 1972 Baxter showed that the general anisotropic XYZ spin-chain Hamiltonian could be obtained as a logarithmic derivative of the transfer matrix of the eight-vertex model, and calculated the ground-state energy for the XYZ model [4]. Subsequently, by constructing the eigenvectors and finding the eigenvalues of the transfer matrix, he solved completely the XYZ model [5]. On the basis of Baxter's results on the transfer matrix, Johnson *et al* calculated the excitation energy of the XYZ model [6]. In 1979 Faddeev proposed the quantum method of inverse problem and applied it to the XYZ model such that the Bethe's ansatz method [8, 9] has been simplified and algebraicised. The results of the transfer matrix were also used to investigate the excitation spectrum of the $\mathbb{Z}_n \times \mathbb{Z}_n$ model originating from the Toda chain [7].

Recently Belavin's $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric model [10–12] was exactly solved [13, 14]. This solution stimulated the investigation of the corresponding 1D quantum chains. In this paper we deduce the Hamiltonian of this model and discuss its characterestics and the relations to other models [7, 15–18].

2. Derivation of the Hamiltonian

The Boltzmann weights $S_{ij}^{kl}(z)$ of Belavin's $\mathbb{Z}_n \times \mathbb{Z}_n$ model can be written as [10–12]

$$S_{ij}^{kl}(z) = \sum_{\alpha \in G_n} w_\alpha(z)(I_\alpha)_{ik}(I_\alpha^\dagger)_{jl} \tag{1}$$

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with

$$w_\alpha(z) = w_\alpha(z, w, \tau) = \rho(z)\vartheta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_1}{n}\\ \frac{1}{2}+\frac{\alpha_2}{n}\end{bmatrix}\left(z+\frac{w}{n}, \tau\right)\left(\vartheta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_1}{n}\\ \frac{1}{2}+\frac{\alpha_2}{n}\end{bmatrix}\left(\frac{w}{n}, \tau\right)\right)^{-1} \tag{2}$$

where $\rho(z)$ is an overall factor which does not change the Yang–Baxter relations [16, 19], $\alpha = (\alpha_1, \alpha_2) \in G_n$, $G_n = \mathbb{Z}_n \times \mathbb{Z}_n$, $I_\alpha = h^{\alpha_1} g^{\alpha_2}$, h and g are $n \times n$ matrices with the matrix elements

$$h_{jk} = \delta^{k+1}_{j(\mathrm{mod}\ n)} \qquad g_{jk} = \omega^k \delta_{jk} \qquad \omega = \exp(2\pi\mathrm{i}/n) \tag{3}$$

and $\vartheta[{}^a_b](z, \tau)$ is the Jacobi theta function of rational characteristics a, b:

$$\begin{aligned}\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z, \tau) &= \sum_{m=-\infty}^{\infty} \exp[\pi\mathrm{i}(m+a)^2\tau + 2\pi\mathrm{i}(m+a)(z+b)]\\ &= \exp[\pi\mathrm{i}a^2\tau + 2\pi\mathrm{i}a(z+b)]\vartheta(z+a\tau+b, \tau).\end{aligned} \tag{4}$$

The Boltzmann weights have been evaluated in [12]. From (1) and (2) it is derived that

$$\begin{aligned}S^{kl}_{ij}(z) = {}& -n\rho(z)\delta_{i+j,k+l}\frac{\vartheta_1'(0, n\tau)}{\vartheta_0'(0, \tau)}\vartheta_1(z, \tau)\vartheta\begin{bmatrix}\frac{1}{2}+\frac{l-k}{n}\\ \frac{1}{2}\end{bmatrix}(z+w, n\tau)\\ &\times\left(\vartheta\begin{bmatrix}\frac{1}{2}+\frac{l-i}{n}\\ \frac{1}{2}\end{bmatrix}(z, n\tau)\vartheta\begin{bmatrix}\frac{1}{2}+\frac{i-k}{n}\\ \frac{1}{2}\end{bmatrix}(w, n\tau)\right)^{-1}\end{aligned} \tag{5}$$

where $\vartheta_1'(0, \tau) = (\partial/\partial z)\vartheta_1(z, \tau)|_{z=0}$. The transfer matrix is

$$T^{l_1 \ldots l_N}_{j_1 \ldots j_N}(z) = \sum_{i_1 \ldots i_N} S^{i_2 l_1}_{i_1 j_1}(z) S^{i_3 l_2}_{i_2 j_2}(z) \ldots S^{i_N l_{N-1}}_{i_{N-1} j_{N-1}}(z) S^{i_1 l_N}_{i_N j_N}(z) \tag{6}$$

satisfying [13, 14]

$$[T(z), T(z')] = 0. \tag{7}$$

It gives many integrals of motion, each of which can be regarded as a Hamiltonian for a quantum system [1]. The simplest one is the Hamiltonian of a multicomponent 1D quantum chain:

$$H = -\rho(0)\frac{\partial}{\partial z}\ln T(z)|_{z=0} + h_0. \tag{8}$$

In the following we evaluate the expression of this Hamiltonian. Using

$$\sum_{\alpha\in G_n} (I_\alpha)_{ik}(I^+_\alpha)_{jl} = n\delta_{il}\delta_{jk} \tag{9}$$

we have, for an N-site 1D chain with periodicity that the $(N+1)$th site identifies the first site

$$H = -\frac{1}{n}\sum_{m=1}^{N} \delta^{l_1}_{j_1} \ldots \delta^{l_{m-1}}_{j_{m-1}} \frac{\partial}{\partial z} S^{l_{m+1} l_m}_{j_m\ j_{m+1}}(z)|_{z=0}\delta^{l_{m+2}}_{j_{m+2}} \ldots \delta^{l_N}_{j_N} + h_0. \tag{10}$$

As contrasted to the w_α in (1), we define Boltzmann coordinates P_α by

$$\sum_{\alpha\in G_n} P_\alpha(z)(I_\alpha)_{il}(I_\alpha^\dagger)_{jk} = \sum_{\alpha\in G_n} w_\alpha(z)(I_\alpha)_{ik}(I_\alpha^\dagger)_{jl}. \tag{11}$$

By means of

$$\operatorname{Tr} I_\alpha I_\beta^\dagger = n\delta_{\alpha\beta} \tag{12}$$

we can get

$$\begin{aligned} P_\alpha(z) &= \frac{1}{n}\sum_{\beta\in G_n} w_\beta(z)\omega^{\alpha_1\beta_2-\alpha_2\beta_1} \\ &= \frac{1}{n}\sum_{c=0}^{n-1}\omega^{-c\alpha_2}S_{0,-c-\alpha_1}^{-c,-\alpha_1}(z) \\ &= -\rho(z)\frac{\vartheta_1'(0,n\tau)}{\vartheta_1'(0,\tau)}\vartheta_1(z,\tau)\left(\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}\end{bmatrix}(z,n\tau)\right)^{-1}F(w) \end{aligned} \tag{13}$$

with

$$F(w)=F(z,w,\tau)=\sum_{c=0}^{n-1}\omega^{-c\alpha_2}\vartheta\begin{bmatrix}\frac{1}{2}+\frac{c-\alpha_1}{n}\\ \frac{1}{2}\end{bmatrix}(z+w,n\tau)\left(\vartheta\begin{bmatrix}\frac{1}{2}+\frac{c}{n}\\ \frac{1}{2}\end{bmatrix}(w,n\tau)\right)^{-1}. \tag{14}$$

This sum can be performed as follows. From

$$F(w+1)=\exp(-2\mathrm{i}\pi\alpha_1/n)F(w) \tag{15}$$

$$F(w+\tau)=\exp[-2\mathrm{i}\pi(z-\alpha_2)/n]F(w) \tag{16}$$

we know that $F(w)$ has only one zero and one pole in the lattice $\Lambda_{1\tau}$ generated by 1 and τ. The pole locates at 0 $(\mathrm{mod}(m+m'\tau))$ and the zero locates at

$$\oint_{\Lambda_{1\tau}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}w\frac{F'(w)}{F(w)}=(\alpha_1\tau+\alpha_2-z)/n \qquad \mathrm{mod}(m+m'\tau). \tag{17}$$

Hence we have

$$F(w)=C_{\alpha_1\alpha_2}(z,w)\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}-\frac{\alpha_2}{n}\end{bmatrix}\left(w+\frac{z}{n},\tau\right)(-\vartheta_1(w,\tau))^{-1}. \tag{18}$$

$C_{\alpha_1\alpha_2}(z,w)$, being a doubly periodic function of w without pole, is independent of w and can be determined to be

$$C_{\alpha_1\alpha_2}(z,0)=\frac{\vartheta_1'(0,\tau)}{\vartheta_1'(0,n\tau)}\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}\end{bmatrix}(z,n\tau)\left(\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}-\frac{\alpha_2}{n}\end{bmatrix}\left(\frac{z}{n},\tau\right)\right)^{-1}. \tag{19}$$

Now the Hamiltonian can be expressed in the 1D chain form with site number summed from 1 to N:

$$H(w,\tau) = -\frac{1}{n}\sum_{m=1}^{N}\sum_{\alpha\in G_n} J_\alpha(w,\tau) I_\alpha^{(m)} I_\alpha^{\dagger(m+1)} + h_0 \tag{20}$$

where

$$J_\alpha(w,\tau) = \frac{\partial}{\partial z}\rho(z)\frac{\vartheta_1(z,\tau)}{\vartheta_1(w,\tau)}\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}-\frac{\alpha_2}{n}\end{bmatrix}\left(\frac{z}{n}+w,\tau\right)\left(\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}-\frac{\alpha_2}{n}\end{bmatrix}\left(\frac{z}{n},\tau\right)\right)^{-1}\Bigg|_{z=0} \tag{21}$$

or

$$J_{00}(w,\tau) = n\rho'(0) + \rho(0)\vartheta_1'(w,\tau)/\vartheta_1(w,\tau) \tag{22}$$

$$J_{\alpha_1\alpha_2}(w,\tau) = \rho(0)\frac{\vartheta_0'(0,\tau)}{\vartheta_1(w,\tau)}\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}-\frac{\alpha_2}{n}\end{bmatrix}(w,\tau)\left(\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}-\frac{\alpha_2}{n}\end{bmatrix}(0,\tau)\right)^{-1} \qquad \alpha\neq(0,0). \tag{23}$$

Clearly the Hamiltonian involves n^2 coupling constants, one of which can become zero if choosing $h_0 = NJ_{00}/n$. The Hamiltonian (20) may be regarded as a multicomponent generalisation of the XYZ model. For $n=2$, taking

$$\rho(z) = 2\pi^{-1}\vartheta_2^{-2}(0,\tau)\vartheta_1(w,2\tau)/\vartheta_4(z+w,2\tau) \qquad h_0 = NJ_{00}/2 \tag{24}$$

we obtain

$$H = -\frac{1}{2}\sum_{m=1}^{N}\left(J_1\sigma_1^{(m)}\sigma_1^{(m+1)} + J_2\sigma_2^{(m)}\sigma_2^{(m+1)} + J_3\sigma_3^{(m)}\sigma_3^{(m+1)}\right) \tag{25}$$

with

$$J_1 = J_{10} = 1 + k\,\mathrm{sn}^2(\tilde{w},k) \qquad J_2 = J_{11} = 1 - k\,\mathrm{sn}^2(\tilde{w},k) \qquad J_3 = J_{01} = \mathrm{cn}(\tilde{w},k)\,\mathrm{dn}(\tilde{w},k) \tag{26}$$

where the modulus of elliptic functions $k = \vartheta_2^2(0,2\tau)/\vartheta_3^2(0,2\tau)$, $\tilde{w} = 2Kw$, $K = \frac{1}{2}\pi\vartheta_3^2(0,2\tau)$. This is the Hamiltonian of the XYZ model and identifies the result of [1, 4].

Using the basis of matrices E_{ij} with $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, the Hamiltonian (20) becomes

$$H(w,\tau) = -\sum_{m=1}^{N}\sum_{ijkl} J_{ijkl}(w,\tau) E_{ij}^{(m)} E_{kl}^{(m+1)} + h_0 \tag{27}$$

with

$$J_{ikjl}(w,\tau) = \frac{1}{n}\frac{\partial}{\partial z} S_{ij}^{lk}(z,w,\tau)\Bigg|_{z=0}. \tag{28}$$

3. The limit $\tau \to \mathrm{i}\infty$ case

We know that the six-vertex model is the special case of the eight-vertex model with the seventh and the eighth vertices vanishing [16]. We shall see below that this can be arrived at by taking the limit $\tau \to \mathrm{i}\infty$ in Belavin's parametrisation. First we consider the limit in the arbitrary n case. From (4), (22) and (23) we get

$$J_{00}(w, \mathrm{i}\infty) = n\rho'(0) + \pi\rho(0)/\sin \pi w \tag{29}$$

$$J_{0\alpha_2}(w, \mathrm{i}\infty) = [\pi\rho(0)/\sin \pi w](\cos \pi w - \cot(\pi\alpha_2/n) \sin \pi w) \qquad \alpha_2 \neq 0 \tag{30}$$

$$J_{\alpha_1\alpha_2}(w, \mathrm{i}\infty) = [\pi\rho(0)/\sin \pi w] \exp(\mathrm{i}\pi w \Delta_{\alpha_1}) \qquad \alpha_1 \neq 0 \tag{31}$$

where

$$\Delta_k = \operatorname{sgn}(k) - 2k/n \qquad k \neq 0. \tag{32}$$

We can see that the coupling constants $J_{0\alpha_2}(w, \mathrm{i}\infty)$ are non-degenerate, $J_{\alpha_1\alpha_2}(w, \mathrm{i}\infty)$ for $\alpha_1 \neq 0$ are n-fold degenerate and there are at most $2n-1$ different coupling constants. For $n=2$ with ρ as in (24) it becomes the XXZ model with

$$J_1 = J_2 = 1 \qquad J_3 = \cos \pi w. \tag{33}$$

Alternatively, if we choose ρ satifying $\rho(z, w, \mathrm{i}\infty) = \lambda\pi^{-1} \sin \pi w$, then from (5) we have the non-vanishing Boltzmann weights:

$$S_{00}^{00}(z, w, \mathrm{i}\infty) = \lambda n \pi^{-1} \sin \pi(z+w) \tag{34}$$

$$S_{kl}^{kl}(z, w, \mathrm{i}\infty) = \lambda n \pi^{-1} \sin \pi z \exp(-\mathrm{i}\pi w \Delta_{l-k}) \tag{35}$$

$$S_{lk}^{kl}(z, w, \mathrm{i}\infty) = \lambda n \pi^{-1} \sin \pi w \exp(i\pi z \Delta_{k-l}). \tag{36}$$

We see that for $w = \mathrm{i}\pi^{-1}\gamma$, $z = \mathrm{i}\pi^{-1}\beta$; $\lambda, \gamma, \beta > 0$, $\rho = \lambda\pi^{-1} \sinh \gamma$, the weights are positive. However, we shall find in the next section that the corresponding Hamiltonian is non-Hermitian in general. From (34)-(36) we get the coupling constants in (27):

$$J_{kkkk}(w, \mathrm{i}\infty) = \lambda \cos \pi w \tag{37}$$

$$J_{ljjl}(w, \mathrm{i}\infty) = \lambda \exp(-\mathrm{i}\pi w \Delta_{j-l}) \qquad j \neq l \tag{38}$$

$$J_{lljj}(w, \mathrm{i}\infty) = \lambda \mathrm{i} \Delta_{j-l} \sin \pi w \qquad j \neq l. \tag{39}$$

Hence (27) becomes

$$H(w, \mathrm{i}\infty) = -\sum_{m=1}^{N} \left(\cos \pi w \sum_j E_{jj}^{(m)} E_{jj}^{(m+1)} \right.$$
$$\left. + \mathrm{i} \sin \pi w \sum_{j \neq l} \Delta_{j-l} E_{ll}^{(m)} E_{jj}^{(m+1)} + \sum_{j \neq l} \exp(-\mathrm{i}\pi w \Delta_{j-l}) E_{lj}^{(m)} E_{jl}^{(m+1)} \right). \tag{40}$$

Letting $w = \mathrm{i}\gamma/\pi$ we have

$$H = -\sum_{m=1}^{N} \left(\sum_{j \neq l} \exp(\gamma \Delta_{j-l}) E_{lj}^{(m)} E_{jl}^{(m+1)} \right.$$
$$\left. + \cosh \gamma \sum_j E_{jj}^{(m)} E_{jj}^{(m+1)} + \sinh \gamma \sum_{j \neq l} \Delta_{j-l} E_{jj}^{(m)} E_{ll}^{(m+1)} \right). \tag{41}$$

4. On the hermiticity of the Hamiltonian

The hermiticity of the Hamiltonian (20) or (27) $H^{\dagger}=H$ is equivalent to the conditions (let $h_0=0$ for simplicity below)

$$J_{\alpha}(w,\tau)^{*}=J_{-\alpha}(w,\tau) \tag{42}$$

or

$$J_{ikjl}(w,\tau)^{*}=J_{kilj}(w,\tau). \tag{43}$$

For (40) we see that (43) leads to

$$\begin{aligned}
&\cos \pi w^{*}=\cos \pi w\\
&\sin \pi w^{*}=-\sin \pi w\\
&\exp(\mathrm{i}\pi w^{*}\Delta_{j-l})=\exp(\mathrm{i}\pi w\Delta_{j-l}).
\end{aligned} \tag{44}$$

For $n>2$ this set of conditions only has the solution $w=$ integer. Hence we know that the Hamiltonians (40) and (20) are not Hermitian for $n>2$ with $w\neq$ integer. For $w=k$ (an integer), choosing ρ satifying $\rho'(0)=0$, $\rho(0)\vartheta_1'(0,\tau)/\vartheta_1(w,\tau)|_{w\to k}=(-1)^k\lambda$, $\lambda^*=\lambda$, we get from (20)-(23) the Hermitian Hamiltonian:

$$H=-\frac{\lambda}{n}\sum_{m=1}^{N}\sum_{\alpha\in G_n}\exp(-2\mathrm{i}\pi k\alpha_1/n)I_{\alpha}^{(m)}I_{\alpha}^{\dagger(m+1)} \tag{45}$$

or

$$H=-\lambda\sum_{m=1}^{N}\sum_{j,l=0}^{n-1}\exp[-2\mathrm{i}\pi k(l-j)/n]E_{lj}^{(m)}E_{jl}^{(m+1)}. \tag{46}$$

It may be referred to as a n-state model [16]. In particular, if $k=n$, it is a generalisation of the XXX model and is investigated by Sutherland [17] and Kulish [18].

5. Discussions

We have deduced the 1D chain Hamiltonian, being the logarithmic derivative of the transfer matrix for Belavin's $\mathbb{Z}_n\times\mathbb{Z}_n$ model. For $n=2$ it is exactly the XYZ model, or the XXZ model for $\tau\to\mathrm{i}\infty$, or the XXX model for $w=n$.

For $n>2$, it can be regarded as a multicomponent generalisation of the above models. However the Hamiltonian for $w\neq$ integer is not Hermitian. This may be a consequence of the non-positivity of the Boltzmann weights shown in [12]. A remarkable fact is that there is a domain of parameters in which the Boltzmann weights (34)-(36) are positive, but the corresponding Hamiltonian is not Hermitian. Positive Boltzmann weights mean that it is a physical system. Though a non-Hermitian Hamiltonian may also be a evolution operator of a physical system (such as a non-stable state, excited atoms or a system described by an optical potential, etc), the physical meaning and the characteristic of the Hamiltonian (41) would still be interesting and could be investigated by means of the solution of the transfer matrix [13, 14].

Another remarkable character is that the gauge transformations of the monodromy matrix may change the coupling constants, but in reality do not change the whole Hamiltonian. To see this, we write the transformed operators by adding a wave. Under the local transformation $G^{(m)}$, the Boltzmann weights become

$$\tilde{S}_{ij}^{kl}(z)^{(m)}=G_{ir}^{(m)}(z)S_{rj}^{sl}(z)G_{sk}^{(m+1)}(z)^{-1} \tag{47}$$

where $G^{(N+1)}=G^{(1)}$, $G_{jk}(0)=\delta_{jk}$, and the G are invertible but may be non-unitary. This transformation ensures the transfer matrix to be invariant, i.e. $\tilde{T}=T$. From (28) and (5) we get $\tilde{J}=J+\Delta J$ with

$$\Delta J^{(m)}_{ikjl}=\rho(0)\left(\frac{\partial}{\partial z}G^{(m)}_{ik}(z)\delta_{jl}+\delta_{ik}\frac{\partial}{\partial z}G^{(m+1)}_{jl}(z)^{-1}\right)_{z=0}. \tag{48}$$

Since $\tilde{T}=T$, or more directly $(\partial/\partial z)P^{(m)-1}P^{(m)}=0$, we have the expected result

$$\Delta H=-\sum_{m=1}^{N}\sum_{ikjl}\Delta J^{(m)}_{ikjl}E^{(m)}_{ik}E^{(m+1)}_{jl}=0. \tag{49}$$

Now let us consider the relation of the Hamiltonian to other two $\mathbb{Z}_n\times\mathbb{Z}_n$ models given in [7, 15]. For [15] the Hamiltonian can be written as

$$\begin{aligned}H_s=-\sum_{m=1}^{N}\Bigg(&\sum_{\sigma\neq\rho}G_{\sigma\rho}E^{(m)}_{\rho\sigma}E^{(m+1)}_{\sigma\rho}+\cosh\eta\sum_{\rho}\varepsilon_\rho E^{(m)}_{\rho\rho}E^{(m+1)}_{\rho\rho}\\&+\sinh\eta\sum_{\sigma\neq\rho}\operatorname{sgn}(\sigma-\rho)E^{(m)}_{\sigma\sigma}E^{(m+1)}_{\rho\rho}\Bigg).\end{aligned} \tag{50}$$

In [7] the Hamiltonian is

$$H_d=-\sum_{m=1}^{N}\left(\sum_{r\neq s}E^{(m)}_{rs}E^{(m+1)}_{sr}+\cosh\gamma\sum_{r}E^{(m)}_{rr}E^{(m+1)}_{rr}+\sinh\gamma\sum_{r\neq s}\Delta_{r-s}E^{(m)}_{rr}E^{(m+1)}_{ss}\right). \tag{51}$$

The gauge transformation $G^{(m)}_{rs}=\delta_{rs}\exp(2s\theta/n)$ changes the coupling constant $\sinh\gamma\Delta_{r-s}$ in (51) to $\sinh\gamma\operatorname{sgn}(r-s)$, and the difference is

$$\begin{aligned}\Delta H&=\sinh\gamma\sum_{m}\sum_{r\neq s}[\Delta_{r-s}-\operatorname{sgn}(r-s)]E^{(m)}_{rr}E^{(m+1)}_{ss}\\&=-\frac{2}{n}\sinh\gamma\sum_{m=1}^{N}\sum_{r_1r_2\ldots r_N}(r_m-r_{m+1})E_{r_1r_1}E_{r_2r_2}\ldots E_{r_Nr_N}=0.\end{aligned} \tag{52}$$

Thus, if we take $\varepsilon_\rho=1$, $G_{\rho\sigma}=1$ or $G_{\rho\sigma}=\exp(\gamma\Delta_{\rho-\sigma})$ in (50), then the Hamiltonian H_s (50) identifies H_d (51) or H (41). We also see that H (41) is different from H_d only in the coupling constants of the $E_{rs}E_{sr}$ terms. However, this difference is very crucial so that the presence of both the coupling constants $e^{\gamma\Delta}$ and $\Delta\sinh\gamma$ in (41) leads to its non-Hermiticity for the $n>2$, $i\gamma\neq$ integer case. Because H (41) in this case is not a normal matrix, it cannot therefore turn into a Hermitian one by a unitary transformation. Moreover, we have known that H (41) has some imaginary eigenvalues in this case. Hence it cannot turn into a Hermitian one by a similarity transformation.

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References

[1] Takhtadzhan L A and Faddeev L D 1979 *Russ. Math. Surveys* **34** 11
[2] Thacker H B 1981 *Rev. Mod. Phys.* **53** 253
[3] Sutherland B 1970 *J. Math. Phys.* **11** 3183
[4] Baxter R J 1972 *Ann. Phys., NY* **70** 323
[5] Baxter R J 1973 *Ann. Phys., NY* **76** 1, 25, 48
[6] Johnson J D, Krinsky S and McCoy B M 1973 *Phys. Rev.* A **8** 2526
[7] Babelon O, de Vega H J and Viallet C M 1982 *Nucl. Phys.* B **200** [FS 4] 266
[8] Bethe H A 1931 *Z. Phys.* **71** 205
[9] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321, 327, **151**, 258
[10] Belavin A A 1981 *Nucl. Phys.* B **180** [FS 2] 189
[11] Tracy C A 1985 *Physica* **16D** 203
[12] Richey M P and Tracy C A 1986 *J. Stat. Phys.* **42** 311
[13] Zhou Y K, Yan M L and Hou B Y 1988 *J. Phys. A: Math. Gen.* **21** L929
[14] Hou B Y, Yan M L and Zhou Y K 1989 *Nucl. Phys.* to appear
[15] Schultz C L 1983 *Physica* **122A** 71
[16] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[17] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[18] Kulish P P and Reshetikhin N Yu 1981 *Zh. Eksp. Teor. Fiz.* **80** 214
[19] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312